Flow-distributed oscillation, flow-velocity modulation, and resonance

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We examine the effects of a periodically varying flow velocity on the standing- and traveling-wave patterns formed by the flow-distributed oscillation mechanism. In the kinematic (or diffusionless) limit, the phase fronts undergo a simple, spatiotemporally periodic longitudinal displacement. On the other hand, when the diffusion is significant, periodic modulation of the velocity can disrupt the wave pattern, giving rise in the downstream region to traveling waves whose frequency is a rational multiple of the velocity perturbation frequency. We observe frequency locking at ratios of 1:1, 2:1, and 3:1, depending on the amplitude and frequency of the velocity modulation. This phenomenon can be viewed as a novel, rather subtle type of resonant forcing.

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Interest has been mounting recently in mechanisms of pattern formation in open reactive flows. The combination of reaction, advection, and diffusion, together with the effect of an upstream boundary condition, leads to mechanisms such as flow-distributed oscillations (FDO) [1–9], a general category of stationary patterns referred to as "flow and diffusion-distributed structures" (FDS) [12–16], and the differential flow instability (DIFI) [10–16]. Our focus here is on FDO. Due to the equivalence [17–19] of flow in reaction-advection-diffusion (RAD) systems and linear growth of the spatial domain of a reaction-diffusion system, and the existence of cellular oscillations in segmenting tissue, FDO was shown to be involved in the axial segmentation occurring during biological development [17–19]. Given the pulsating growth of certain organisms [22,23], including human embryos [24], we study here the consequences of a periodically modulated flow $v(t)$ on FDO.

The systems of interest are described by the RAD equation without differential transport,

$$\frac{\partial \mathbf{U}}{\partial t} = \mathbf{f}(\mathbf{U}) - v(t)\frac{\partial \mathbf{U}}{\partial x} + D\frac{\partial^2 \mathbf{U}}{\partial x^2}, \tag{1}$$

where D is the diffusion constant, $v(t)$ is the flow velocity, $\mathbf{U}(x,t)$ is an N-dimensional vector of dynamical variables (concentrations of species), and the local dynamics given by the vector valued rate function $\mathbf{f}(\mathbf{U})$ has an attracting limit cycle. If $v(t)$ is constant, this system can support FDO controlled by the upstream boundary condition $\mathbf{U}(0,t)$. In the simplest case, a constant boundary condition sets the phase of each oscillating fluid element as it enters the medium, and the periodic recurrence of the same phase as the fluid travels downstream results in stationary waves. Oscillating boundary conditions result in traveling waves [17–19,5]. Diffusion can modify the effective dynamics of the medium as it travels downstream and even extinguish the oscillations [7,9]. Equation (1) is also relevant to media such as linearly growing organisms, as it can be reinterpreted by means of a Galilean transformation as representing a stationary medium with a boundary (the growth tip) moving at speed $v(t)$ [17–19,21].

We examine the effect of a sinusoidally varying velocity $v(t)=v_0+\delta_v \cos \omega_v t$. In the *kinematic* limit of vanishing diffusion $D/v^2 \to 0$, the wave pattern undergoes a simple, calculable longitudinal displacement which is periodic in both time and space. Away from the kinematic limit, however, we observe a type of nonlinear resonance. Relatively small disturbances of an FDO wave pattern are magnified with downstream distance until the wavefronts break. This rupture generates traveling waves in the downstream region whose temporal frequency is a rational multiple of the velocity perturbation frequency. We observe 1:1, 2:1, and 3:1 ratios depending on the frequency and amplitude of the velocity perturbation.

For the numerical examples, we use the FitzHugh-Nagumo-type (FN) dynamics [25]

$$\mathbf{f}(X,Y) = \begin{pmatrix} \varepsilon(X - X^3 - Y) \\ -Y + \alpha X + \beta \end{pmatrix} \tag{2}$$

with $\varepsilon=5$, $\alpha=2$, and $\beta=0$ for the local rate function. (This gives a generic nonlinear oscillator.) At these parameter values, the local system has a limit cycle and a moderately strong nonlinearity. The frequency of the limit cycle oscillation is $\omega_0 \approx 2\pi(0.43)$. In all simulations, we set $D=1$ and vary only v_0 and δ_v.

When the ratio D/v^2 is sufficiently small, diffusion is relatively unimportant and each individual fluid element behaves approximately as an independent oscillator obeying the local dynamics whose initial phase is set by the boundary condition as it enters the flow from the upstream end [3]. The phase fronts can then be calculated by pure kinematics: the oscillation phase of a fluid element at a particular time and location depends on its initial phase when it entered the flow and how long ago it entered the flow. For the case of a stationary boundary condition (i.e., constant phase at the boundary), the result is that the location of the phase front for a particular value of the oscillation phase ϕ is given by

$$x(\phi,t) = v_0\frac{\phi}{\omega_0} + \frac{\delta_v}{\omega_v}\left[\sin \omega_v t - \sin \omega_v\left(t - \frac{\phi}{\omega_0}\right)\right], \tag{3}$$

where ω_0 is the frequency of the local oscillator. When $\delta_v=0$, this reduces to the simple linear mapping between position and phase that characterizes stationary FDO waves.

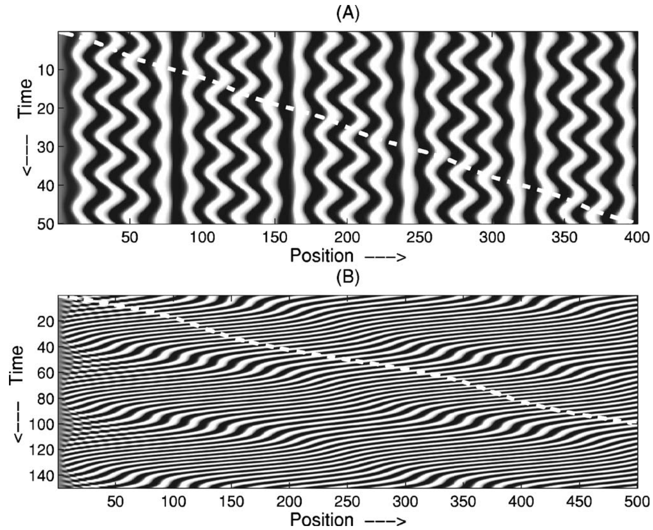

FIG. 1. Periodic modulation of standing and traveling FDO waves in the kinematic limit. Note the periodicity in both time and space. In each space-time diagram, the trajectory of a comoving point is shown as a guide to the eye (dashed white line). (A) Stationary waves: $v_0=8$, $\delta_v=2$, $\omega_v=2\pi(0.1)\approx\omega_0/4$. Some wavefronts remain stationary while others wiggle back and forth. (B) Upstream traveling waves with wave frequency $\omega_{\rm tw}=2\pi(0.25)$ subject to a modulated velocity field with $v_0=5$, $\delta_v=2$, $\omega_v=2\pi(0.02)$.

Note the following: (i) The amplitude of the displacement of the phase fronts [second, time-dependent term in Eq. (3)] depends on δ_v/ω_v, implying that faster (higher frequency) velocity modulation has less of an effect than slower modulation; (ii) the displacement is periodic in time with the same frequency as the velocity perturbation; but (iii) it is also periodic in ϕ and thus in space. The fronts for which ϕ is a multiple of $\pi\omega_0/\omega_v$ are not displaced, and they occur periodically at positions $\pi n v_0/\omega_v$. The spatial periodicity can be understood by considering the trajectories of fluid elements entering the system at different times. Different elements enter at different points in the velocity modulation period and thus begin their downstream travel at different initial velocities. Over any multiple of the modulation period, however, the velocity averages to v_0 and thus all elements reach the same position at the same phase when one full period has passed, regardless of when they started.

When the boundary condition is oscillatory instead of stationary, traveling waves are generated. Just as in the stationary case, velocity modulation causes a periodic longitudinal displacement of the traveling wavefronts. The modulation of stationary and traveling waves is illustrated in Fig. 1.

When diffusion is unimportant, neighboring fluid elements do not interact and the behavior of FDO patterns can be explained by pure kinematics. Each comoving fluid element follows the limit cycle defined by the batch reactor dynamics. However, significant diffusion alters the dynamics. The flow velocity modulation then introduces a periodic variation in the local environment of each fluid element. The strength of the local gradient is different for each comoving element and diffusion therefore affects the dynamics differently at different locations. This differential effect can magnify the small kinematic effect of the velocity perturbation, leading to larger differences in the dynamical variables and eventually to disruption of the smooth FDO waveforms.

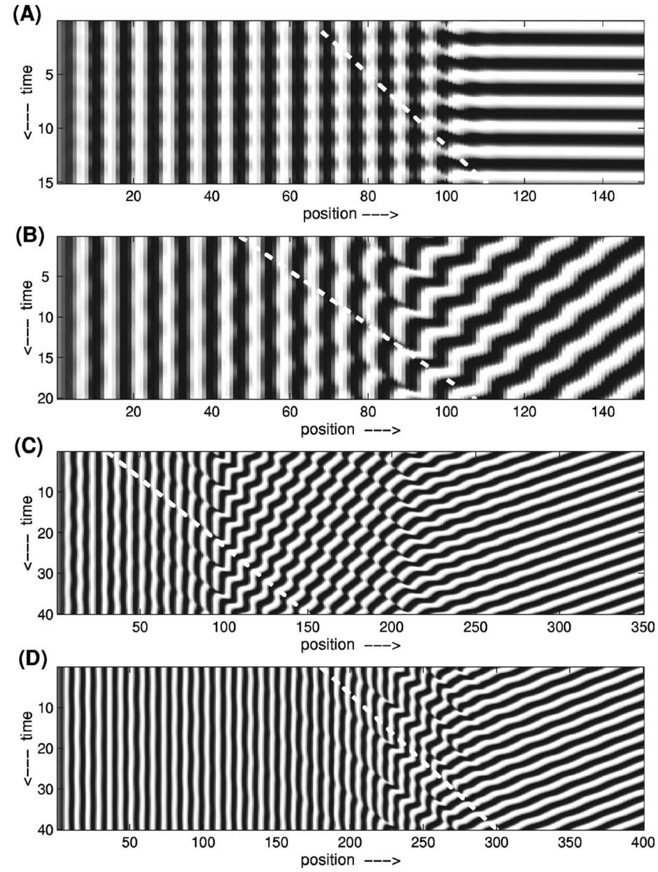

FIG. 2. Examples of the breakup of waves due to a velocity perturbation. All examples have average flow velocity $v_0=3$. The dotted white line in each frame represents v_0. (A) $\omega_v=2\pi(0.43)\approx\omega_0$, $\delta_v=0.215$. A slight pulsation with frequency ω_v is visible in the standing wavefronts, becoming more pronounced downstream. Near $x=100$, there is a transition to uniform oscillation. (B) $\omega_v=2\pi(0.2)\approx\omega_0/2$, $\delta_v=0.1$. Periodic disturbance of the standing waves becomes sharper with increasing distance, and there is a transition to traveling waves with a 1:1 frequency ratio. Near the transition, these traveling waves propagate in a saltatory manner but they grow smoother with further downstream distance. (C) $\omega_v=2\pi(0.15)\approx\omega_0/3$, $\delta_v=0.1125$. As in (B), there is a transition to traveling waves near $x=100$, but in this case the waves do not smooth out with downstream distance. Instead, there is a second transition at $x=200$ to waves with twice the velocity modulation frequency (2:1 resonance). (D) $\omega_v=2\pi(0.10)\approx\omega_0/4$, $\delta_v=0.05$. A series of transitions leads to waves with three times the perturbation frequency (3:1 resonance).

Figure 2 shows several examples of this phenomenon, in which quite subtle modulations of a stationary wave pattern become magnified with increasing downstream distance and lead to the breaking and reconnection of wavefronts in the downstream region. The simulations in these examples were all done at an average flow velocity of $v_0=3$. By comparison, the boundary between absolute and convective instability of the Hopf/FDO instability occurs at $v_{AC}\approx 2.82$. Stationary waves controlled by the boundary remain possible at velocities well below this threshold, however [26]. Thus, while the flow velocity is far from the kinematic limit, it lies well within the regime where boundary-controlled stationary waves are stable in the absence of flow modulation. The minimum velocity $v_0 - \delta_v$ never falls below v_{AC} except in Fig. 2(A), and then only by a small amount.

In Fig. 2(A), the perturbation frequency is very close to the natural frequency of the chemical oscillator. The effect of the flow modulation is visible in this space-time plot as a slight pulsation of the wavefronts. The pulsation becomes stronger at positions farther downstream, until there is a transition to a region of nearly uniform synchronous oscillation, synchronized to the period of the flow modulation. If the modulation frequency is changed, the pattern in the downstream region remains synchronized to the modulation, and the result is either upstream or downstream traveling waves. An example of upstream waves is shown in Fig. 2(B). In this case, the velocity modulation is at a frequency lower than the intrinsic natural frequency of the medium. As one can see from the figure, the system's response to the velocity perturbation is nonlinear. Instead of a simple sinusoidal displacement, the stationary wavefronts develop a series of sharp cusps. At a certain downstream position, the wavefronts break and reconnect, and the periodic disturbances become the source of a set of traveling waves with frequency equal to the modulation frequency, just as if the boundary were being driven at that frequency.

Just as in the case of ordinary FDO phase waves driven by a perturbation at the boundary [3–5], perturbations slower than the intrinsic frequency give rise to upstream traveling waves. (In general, the phase velocity, wavelength, and frequency of the traveling waves obey the kinematic relationships discussed in [17–19,5,9].) These waves propagate irregularly in the region just downstream from the transition, but with increasing downstream distance they become smoother. The temporal frequency of the waves is locked to the velocity modulation frequency. Figure 2(C) shows a more complicated situation with two consecutive transitions. The first transition to travelling waves occurs much as in Fig. 2(B). However, instead of smoothing out with downstream distance, these waves propagate irregularly and develop a second instability at a position farther downstream, leading to traveling waves with a temporal frequency exactly *twice* that of the velocity perturbation. This can be viewed as a form of 2:1 frequency locking. The latter traveling wave smooths out with downstream distance and appears to be the final asymptotic waveform. An asymptotic waveform at three times the velocity perturbation frequency is also possible, as in Fig. 2(D). In general, a sequence of transitions leads to successive regions of stationary, 1:1, 2:1, 3:1, etc. waves. In the particular case of Fig. 2(D), the three transitions are quite close together. Which asymptotic waveform is selected, and the exact distances from one transition to the next, depend in nontrivial ways on the perturbation frequency and amplitude. We will explore this dependence in subsequent work; a behavior somewhat analogous to Arnold tongues seems to occur. We have observed asymptotic waves in 1:1, 2:1, and 3:1 ratios to the perturbation frequency, but we have not yet observed other rational multiples such as 2/3. Interestingly, the tendency of stationary waves to break is strongest not at the intrinsic natural frequency of the oscillator, but at approximately $0.65\omega_0$.

Immediately downstream from any transition point, the traveling waves generally propagate with a pulsating phase velocity, but become smoother with increasing distance downstream. Such behavior was observed in both experiments and numerical simulations for waves forced at the boundary under a steady flow velocity [4]. In that case, the pulsating phase velocity was due to a mismatch between the oscillations driving the waves and the limit cycle of the intrinsic dynamics in the flow reactor. The explanation in this case is the same. Instead of being driven by an oscillation at the inflow boundary, however, these traveling waves are driven by an oscillation *induced* by the flow velocity modulation. As in the case without flow modulation, diffusion tends to smooth the jumping waves as they travel downstream, unless the velocity perturbation induces a second instability as in Figs. 2(C) and 2(D).

The spatiotemporal resonance manifested in wavefront disruption and frequency locking is novel. While previous studies of spatiotemporal resonance involved direct, global perturbations of the local dynamics [27,28], in the present case the perturbation acts *only* at the inflow boundary. This becomes more evident when one considers the equivalent growing reaction-diffusion system in the comoving frame [20,19], where the velocity does not enter into the dynamical equations except via the boundary condition. Yet this boundary effect propagates into the spatial domain where it leads the breakup of waves and resonant frequency locking.

Due to the above-mentioned equivalence of flow and growth [18,19], the same phenomenon should be observable in experiments such as those of [20,21], which use a stationary medium with a moving boundary, if the velocity of the boundary is modulated. It may also be relevant to biological situations [22–24] in which growth is pulsatile.

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